# A COMMON FIXED POINT THEOREM ON FUZZY METRIC SPACE

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**ABSTRACT**

The purpose of this paper is to prove a common fixed point theorem in a fuzzy metric space using the concept of weakly compatible mappings.

**Keywords:** Fixed point, self maps, weakly compatible mappings.

**AMS (2000) Mathematics Classification:** 54H25, 47H10

## 1. INTRODUCTION

The concept of the fuzzy metric spaces has been introduced by Zadeh [1], in the recent past many authors developed the theory of fuzzy sets and applications. Grabic [4] obtained the Banach contraction principle in fuzzy version. Recently George and Veeramani [3] have shown that every metric induces a fuzzy metric.

### Definitions and Preliminaries

**Definition 1.1:** A binary operation * :$[0,1] \times [0,1] \rightarrow [0,1]$ is called continuous t-norm if * satisfies the following conditions:

(i)     * is commutative and associative
(ii)    * is continuous
(iii)   a*1=a for all a$\in$[0,1]
(iv)    a*b $\leq$ c*d whenever a $\leq$c and b$\leq$ d for all a, b,c,d $\in$[0,1]

**Definition 1.2:** A 3-tuple (X, M,*) is said to be fuzzy metric space if X is an arbitrary set,* is continuous t- norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for all x,y,z$\in$X, s,t>0

(FM-1)  M(x,y,0)=0
(FM-2)  M(x,y,t)=1 for all t>0 if and only if x=y
(FM-3)  M(x,y,t)= M(y,x,t)
(FM-4)  M(x,y,t) * M(y,z,s) $\leq$ M(x,z,t+s)
(FM-5)  M(x,y,.) : $[0,\infty) \rightarrow [0,1]$ is left continuous

The function M(x, y,t) denote the degree of nearness between x and y with respect to 't'. We observe M(x,y,t)=1 when x=y for all t>0 and M(x,y,t) =0 with t=$\infty$. Some of the properties of fuzzy metric space and examples are given in paper George and Veeramani.

**Example 1.3 (**Induced fuzzy metric space**):** Let (X,d) be a metric space defined a*b=min{a,b} for all x,y$\in$X and t>0,

$$M(x, y, t) = \frac{t}{t + d(x, y)} \qquad\qquad ----(a)$$

Then (X, M,*) is a fuzzy metric space. We call this fuzzy metric $M$ induced by metric $d$ the standard fuzzy metric. From the above example every metric induces a fuzzy metric but there exist no metric on X satisfying (a).

**Definition 1.4**: Let (X, M,*) be a fuzzy metric space then a sequence $<x_n>$ in X is said to be convergent to a point $x \in X$, if
$$\lim_{n \to \infty} M(x_n, x, t) = 1 \quad \text{for all } t > 0.$$

**Definition 1.5**: A sequence $<x_n>$ in X is called a Cauchy sequence if $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \quad \text{for all } t > 0 \text{ and } p > 0.$

**Definition 1.6:** A fuzzy metric space (X, M,*) is said to be complete if every Cauchy sequence is convergent to a point in X.

**Lemma 1.7 ([4]):** For all $x, y \in X$, M(x,y,.) is non decreasing.
**Lemma 1.8 ([5])** let (X,M,*)be a fuzzy metric space if there exists $k \in (0,1)$ such that M(x,y,kt)≥M(x,y,t) then x=y.

**Proposition 1.9 ([3])**: In the fuzzy metric space (X, M,*) if a*a≥a for all $a \in [0,1]$ then a*b= min{a,b}

**Definition 1.10:** Two self maps S and T of a fuzzy metric space (X,M,*) are said to be compatible mappings if $\lim_{n \to \infty} M(STx_n, TSx_n, t) = 1$, whenever $<x_n>$ is a sequence in X such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$ for some $z \in X$.

**Definition 1.11:** Two self maps S and T of a fuzzy metric space (X,M,*) are said to be weakly compatible if they commute at their coincidence point. i.e if Su=Tu for some $u \in X$ then STu=TSu.

It is clear that every compatible pair is weakly compatible but its converse need not be true.

To know the relation among commutatively, compatibility and weakly compatibility refer some of the research papers like Jungck[7], Pant[11], Srinivas and others[12].
**2.1 Theorem:** Let A, B, S and T are self maps of a complete Fuzzy metric space (X,M,*) satisfying the conditions

A(X)⊆T(X) and B(X) ⊆ S(X) .........(2.1.1)
The pairs (A, S) and (B, T) are weakly compatible .......... (2.1.2)
$$[M(Ax, By, kt)]^2 * M(Ax, By, kt)M(Ty, Sx, kt) \geq \{k_1[M(By, Sx, 2kt) * M(Ax, Ty, 2kt)]$$
$$+ k_2[M(Ax, Sx, kt) * M(By, Ty, kt)]\}M(Ty, Sx, t)$$

Where for all x,y in X and $k_1, k_2 \geq 0$, $k_1 + k_2 \geq 1$ .......(2.1.3)
Then A, B, S and T have a unique common fixed point z in X.

**(2.1.4) Lemma:** Let A,B,S and T be self mappings from a complete metric space (X,d) into itself satisfying the conditions (2.1.1)and (2.1.3)Then the sequence $\{y_n\}$ defined by $y_{2n}=Ax_{2n}= Tx_{2n+1}$ and $y_{2n+1}=Bx_{2n+1}= Sx_{2n+2}$ for n ≥ 0 relative to four self maps is a cauchy sequence in X.

**Proof:** From the conditions (2.1.1) (2.1.3) and from the definition of associated sequence we have

Let $x_0$ be any arbitrary point of X, $A(X)\subseteq T(X)$ and $B(X) \subseteq S(X)$ there exists $x_1,x_2\in X$ such that $Ax_0=Tx_1$ and $Bx_1=Sx_2$.

Inductively we construct a sequence $<x_n>$ and $<y_n>$ in X such that $y_{2n}=Ax_{2n}= Tx_{2n+1}$ and $y_{2n+1}=Bx_{2n+1} = Sx_{2n+2}$ for n ≥ 0.

By taking $x= x_{2n}$ $y_= x_{2n+1}$

$$\left[ M(Ax_{2n},Bx_{2n+1},kt)\right]^2 * M(Ax_{2n},Bx_{2n+1},kt)M(Tx_{2n+1},Sx_{2n},kt) \geq \{k_1\left[ M(Bx_{2n+1},Sx_{2n},2kt)*M(Av,Tx_{2n+1}2kt)\right]$$
$$+ k_2\left[ M(Ax_{2n},Sx_{2n},kt)*M(Bx_{2n+1},Tx_{2n+1},kt)\right]\}M(Tx_{2n+1},Sx_{2n},t)$$

$$\left[ M(y_{2n},y_{2n+1},kt)\right]^2 * M(y_{2n},y_{2n+1},kt)M(y_{2n},y_{2n-1},kt) \geq \{k_1\left[ M(y_{2n+1},y_{2n-1},2kt))*M(y_{2n},y_{2n},2kt)\right]$$
$$+ k_2\left[ M(y_{2n},y_{2n-1},kt)*M(y_{2n+1},y_{2n},kt)\right]\}M(y_{2n},y_{2n-1},t)$$

$$\left[ M(y_{2n},y_{2n+1},kt)\right]\{M(y_{2n},y_{2n+1},kt)*M(y_{2n},y_{2n-1},kt)\} \geq \{k_1\left[ M(y_{2n+1},y_{2n-1},2kt)\right]$$
$$+ k_2\left[ M(y_{2n},y_{2n-1},kt)*M(y_{2n+1},y_{2n},kt)\right]\}M(y_{2n},y_{2n-1},t)$$

$$\left[ M(y_{2n},y_{2n+1},kt)\right]\{M(y_{2n+1},y_{2n-1},2kt)\} \geq \{k_1\left[ M(y_{2n+1},y_{2n-1},2kt)\right]+k_2[M(y_{2n-1},y_{2n+1_1},2kt)]\}M(y_{2n},y_{2n-1},t)$$

$$\left[ M(y_{2n},y_{2n+1},kt)\right][M(y_{2n+1},y_{2n-1},2kt)] \geq \{k_1\left[ M(y_{2n+1},y_{2n-1},2kt)\right] +k_2[M(y_{2n-1},y_{2n+1_1},2kt)]\}M(y_{2n},y_{2n-1},t)$$

$$\left[ M(y_{2n},y_{2n+1},kt)\right][M(y_{2n+1},y_{2n-1},2kt)] \geq \{k_1+k_2\}\left[ M(y_{2n+1},y_{2n-1},2kt)\right]M(y_{2n},y_{2n-1},t)$$

$$\left[ M(y_{2n},y_{2n+1},kt)\right] \geq \{k_1+k_2\} M(y_{2n-1},y_{2n},t)$$

$$\left[ M(y_{2n},y_{2n+1},kt)\right] \geq M(y_{2n-1},y_{2n},t)$$

$$\left[ M(y_n,y_{n+1},kt)\right] \geq M(y_{n-1},y_n,t)$$

$$M(y_n,y_{n+1},t) \geq M(y_{n-1},y_n,t/k) \geq M(y_{n-1},y_n,t/k^2) \geq M(y_{n-1},y_n,t/k^3).......\geq M(y_{n-1},y_n,t/k^n)$$

This implies $M(y_n,y_{n+1},t)\rightarrow 1$ as n→∞

For each $\varepsilon >0$ and t>0 we can choose $n_0\in N$ such that $M(y_n,y_{n+1},t)> 1-\varepsilon$

For m,n∈ N suppose m ≥n

$$M(y_n,y_m,t) \geq \left[ M(y_n,y_{n+1},t/{m-n})*M(y_{n+1},y_{n+2},t/{m-n})*....*M(y_{m-1},y_m,t/{m-n})\right]$$
$$\geq (1-\varepsilon)*(1-\varepsilon)*.....(1-\varepsilon)$$
$$\geq (1-\varepsilon)$$

This shows that the sequence $\{y_n\}$ is a cauchy sequence in X and, it converges to a limit, say z∈X.

**Proof of main Theorem:**

$Ax_{2n,} \to z \qquad Tx_{2\,n+1} \to z \qquad , \qquad Bx_{2n+1} \to z, \qquad Tx_{2n+1} \to z$

Since the condition $A(X) \subseteq T(X)$ implies there exists $u \in X$ such that $Tu = z$

From the condition $B(X) \subseteq S(X)$ implies there exists $v \in X$ such that $Sv = z$

To prove $Av = z$

Put $x = v, y = x_{2n+1.}$

$$[M(Av, Bx_{2n+1}, kt)]^2 * M(Av, Bx_{2n+1}, kt)M(Tx_{2n+1}, Sv, kt) \geq \{k_1[M(Bx_{2n+1}, Sv, 2kt) * M(Av, T, x_{2n+1} 2kt)]$$
$$+ k_2[M(Av, Sv, kt) * M(Bx_{2n+1}, Tx_{2n+1}, kt)]\}M(Tx_{2n+1}, Sv, t)$$

$$[M(Av, z, kt)]^2 * M(Av, z, kt)M(z, z, kt) \geq \{k_1[M(z, z, 2kt) * M(Av, z, 2kt)]$$
$$+ k_2[M(Av, z, kt) * M(z, z, kt)]\}M(z, z, t)$$

$$[M(Av, z, kt)]^2 * M(Av, z, kt) \geq \{k_1[M(Av, z, 2kt)]$$
$$+ k_2[M(Av, z, kt)]\}M(z, z, t)$$

$$\{[M(Av, z, kt)][M(Av, z, kt)] * M(Av, z, kt)\} \geq \{k_1[M(Av, z, 2kt)]$$
$$+ k_2[M(Av, z, kt)]\}M(z, z, t)$$

$$\{[M(Av, z, kt)][M(Av, z, kt)]\} \geq \{k_1[M(Av, z, kt)]$$
$$+ k_2[M(Av, z, kt)]\}M(z, z, t)$$

$$[M(Av, z, kt)]^2 \geq \{k_1 + k_2\}[M(Av, z, kt)]$$

$$[M(Av, z, kt)] \geq \{k_1 + k_2\}$$

$$[M(Av, z, kt)] \geq 1$$

$$\therefore Av = z$$

Since, $(k_2 + k_1) \geq 1$, *where* $k_1, k_2 \geq 0$ get $Av = z$. Thus $Av = Sv = z$.

Since the pair $(A,S)$ is weakly compatible and $Av = Sv = z$ implies $ASv = SAv$ or $Az = Sz$.

To prove $Bu = z$, Let $x = x_{2n}$, $y = u$

$$[M(Ax_{2n}, Bu, kt)]^2 * M(Ax_{2n}, Bu, kt)M(Tu, Sx_{2n}, kt) \geq \{k_1[M(Bu, Sx_{2n}, 2kt) * M(Ax_{2n}, Tu, 2kt)]$$
$$+ k_2[M(Ax_{2n}, Sx_{2n}, kt) * M(Bu, Tu, kt)]\}M(Tu, Sx_{2n}, t)$$

$$[M(z, Bu, kt)]^2 * M(z, Bu, kt)M(z, z, kt) \geq \{k_1[M(Bu, z, 2kt) * M(z, z, 2kt)] + k_2[M(z, z, kt) * M(Bu, z, kt)]\}M(z, z, t)$$

$$[M(z, Bu, kt)]^2 * M(z, Bu, kt) \geq \{k_1[M(Bu, z, 2kt)] + k_2[M(Bu, z, kt)]\}$$

$$[M(z, Bu, kt)]\{[M(z, Bu, kt)] * M(z, Bu, kt)\} \geq \{k_1[M(Bu, z, 2kt)] + k_2[M(Bu, z, kt)]\}$$

$$[M(z, Bu, kt)]\{[M(z, Bu, kt)]\} \geq \{k + k_2\}[M(Bu, z, kt)]$$

$$[M(z, Bu, kt)]^2 \geq \{k + k_2\}[M(Bu, z, kt)]$$

$$[M(z, Bu, kt)] \geq \{k + k_2\}$$

$$[M(z, Bu, kt)] \geq 1$$

$$\therefore Bu = z$$

Again since the pair $(B,T)$ is weakly compatible and $Bu = Tu = z$ implies $BTu = TBu$ or $Bz = Tz$.

To prove Az=z   put      x=z,  y=x$_{2n+1}$

$$\left[M(Az,Bx_{2n+1},kt)\right]^2 * M(Az,Bx_{2n+1},kt)M(Tx_{2n+1},Sz,kt) \geq \{k_1\left[M(Bx_{2n+1},Sz,2kt)*M(Az,T,x_{2n+1}2kt)\right]$$
$$+ k_2\left[M(Az,Sz,kt)*M(Bx_{2n+1},Tx_{2n+1},kt)\right]\}M(Tx_{2n+1},Sz,t)$$

$$\left[M(Az,z,kt)\right]^2 * M(Az,z,kt)M(z,Az,kt) \geq \{k_1\left[M(z,Az,2kt)*M(Az,z,2kt)\right]$$
$$+ k_2\left[M(Az,Az,kt)*M(z,z,kt)\right]\}M(z,Az,t)$$

$$\left[M(Az,z,kt)\right] \geq \frac{k_2}{1-k_1}$$
$$\left[M(Az,z,kt)\right] \geq 1$$
$$\therefore Az = z$$

To prove Bz = z  Put x = z, y = z

$$\left[M(Az,Bz,kt)\right]^2 * M(Az,Bz,kt)M(Tz,Sz,kt) \geq \{k_1\left[M(Bz,Sz,2kt)*M(Az,Tz,2kt)\right]$$
$$+ k_2\left[M(Az,Sz,kt)*M(Bz,Tz,kt)\right]\}M(Tz,Sz,t)$$

$$\left[M(z,Bz,kt)\right]^2 * M(z,Bz,kt)M(Bz,z,kt) \geq \{k_1\left[M(Bz,z,2kt)*M(z,Bz,2kt)\right]$$
$$+ k_2\left[M(z,z,kt)*M(Bz,Bz,kt)\right]\}M(Bz,z,t)$$

$$\left[M(z,Bz,kt)\right]^2 * \left[M(z,Bz,kt)\right]^2 \geq \{k_1\left[M(Bz,z,2kt)\right] + k_2\left[1\right]\}M(Bz,z,t)$$

$$\{\left[M(z,Bz,kt)\right]^2\} \geq \{k_1\left[M(Bz,z,2kt)\right] + k_2\}M(Bz,z,kt)$$
$$\{\left[M(z,Bz,kt)\right]\} \geq \{k_1\left[M(Bz,z,kt)\right] + k_2\}$$
$$\left[M(z,Bz,kt)\right]\{1-k_1\} \geq \{k_2\}$$
$$\left[M(z,Bz,kt)\right] \geq \frac{k_2}{1-k_1}$$
$$\left[M(z,Bz,kt)\right] \geq 1$$
$$\therefore Bz = z$$

Since Az=Bz=Sz=Tz=z,we get z is a common fixed point of A,B,S and T. The uniqueness of the fixed point can be easily proved.

**(2.2.5) Example:** Let   X= [0,1],   $M(x,y,t) = \dfrac{t}{t+d(x,y)}$      where  d(x,y)=$|x-y|$

$$Ax = Bx = \begin{cases} \dfrac{1}{8} & if \ 0 \leq x \leq \dfrac{1}{8} \\[2mm] \dfrac{1}{6} & if \ \dfrac{1}{8} < x \leq 1 \end{cases} \qquad Sx = Tx = \begin{cases} \dfrac{1}{4}-x & if \ 0 \leq x \leq \dfrac{1}{8} \\[2mm] \dfrac{1}{4} & if \ \dfrac{1}{8} < x \leq 1 \end{cases}$$

Then A(X) =B(X)=$\left\{\dfrac{1}{8},\dfrac{1}{6}\right\}$  while  S(X) =T(X) =$\left\{\dfrac{1}{4}\cup\left[\dfrac{1}{4},\dfrac{1}{8}\right]\right\}$ so that   the conditions A(X) $\subseteq$ T(X) and  B(X) $\subseteq$ S(X) are  satisfied .

From the example given above, clearly the pairs (A,S) and (B,T) are weakly compatible as they commute at coincident point 1/8 . But the pairs (A,S) and (B,T) are not compatible

For this, take a sequence $x_n = \left( \dfrac{1}{8} - \dfrac{1}{n} \right)$ for n≥1, then $\lim\limits_{n \to \infty} Ax_n = \lim\limits_{n \to \infty} Sx_n = 1/8$ and

$\lim\limits_{n \to \infty} ASx_n = 1/6$ also $\lim\limits_{n \to \infty} SAx_n = 1/8$ So that $\lim\limits_{n \to \infty} M(ASx_n, SAx_n, t) \neq 1$. Also note that none of the mappings are continuous and the rational inequality holds for the values of $0 \le k_1 + k_2 \ge 1$, where $k_1, k_2 \ge 0$. Clearly 1/8 is the unique common fixed point of A, B, S and T.

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